Reasoning about Numbers and Quantities

Course 1 of Reconceptualizing Mathematics for Elementary and Middle School Teachers

Student Version

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Note: The materials in this module were developed at San Diego State University in part with funding from the National Science Foundation Grant No. ESI 9354104. The content of this module is solely the responsibility of the authors and does not necessarily reflect the views of the National Science Foundation.
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All three authors consider themselves as having dual roles—as teacher educators and as researchers on the learning and teaching of mathematics. Most of their research took place in elementary and middle school classrooms and in professional development settings with teachers of these grades.
Reasoning about Numbers and Quantities

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Chapter 1 Reasoning About Quantities

We encounter quantities of many kinds each day. In this chapter you will be asked to think about quantities and the manner in which we use them to better understand our lives. In particular, you will encounter “story problems” and learn how to solve them so that you haven’t done so already.

1.1 What is a Quantity?

Consider the following questions:

How long do humans live?

How fast is the wind blowing?

Which is more crowded, New York City or Mexico City?

How big is this room?

How far is it around the earth?

The answer to each of these questions involves some quantity.

Definitions: A quantity is anything (an object, event, or quality thereof) that can be measured or counted. The value of the quantity is its measure or the number of items that are counted. A value of a quantity involves a number and a unit of measure or number of units.

For example, the length of a room is a quantity. It can be measured. Suppose the measurement is 14 feet. Note that 14 is a number, and feet is a unit of measure: 14 feet is the value of the quantity length of room. For another example: The number of people in the bus is a quantity. Suppose the count is 22 people. Note that 22 is a number, and the unit counted is people, so 22 people is the value associated with the quantity of the number of people on the bus.
A person’s age, the speed of the wind, the population density of New York City, the area of this room, and the distance around the earth at the equator are all examples of quantities.

Think About... What are some possible values of these last quantities? Notice that population alone will not be sufficient to address the question of how crowded a city or a country is. Why?

Not all qualities of objects, events, or persons can be quantified. Consider love. Even though young children, in an attempt to quantify love, are often quick to stretch their arms our wide when asked “How much do you love me?” But love is not a quantity. Love, anger, boredom, and interest are some examples of qualities that are not quantifiable. Feelings, in general, are not quantifiable—thus, they are difficult to assess.

Think About... Name some other “things,” besides feelings, that are not quantities.

It should be clear to you that a quantity is not the same as a number. In fact, one can think of a quantity without knowing its value. For example, the amount of rain fallen on a given day is a quantity, regardless of whether or not someone measured the actual number of inches of rain fallen. One can speak of the amount of rain fallen without knowing how many inches fell. Likewise, one can speak of a dog’s weight, a tank’s capacity, the speed of the wind, the amount of time it takes to do a chore (all quantities) without knowing their actual values.

Discussion: Identifying Quantities and Measures

1. Identify the quantity or quantities that each of the following questions address.
   a. How tall is the Eiffel tower?
   b. How fast does water pour out of a faucet?
   c. Which is wealthier, Honduras or Mozambique?
   d. How much damage did the earthquake cause?

2. Identify an appropriate unit of measure that can be used to determine the value of the quantities involved in answering the
above questions. Is the wealth of a country measured the same way as the wealth of an individual? Explain.

**Activity: Easy To Quantify?**

1. Many attributes or qualities of objects are easily quantifiable. Others are not so straightforward. Of the following, which are easy to quantify and which ones aren’t?
   - a. The weight of a newborn baby
   - b. The gross national product
   - c. Student achievement
   - d. Blood pressure
   - e. Livability of a city
   - f. Infant mortality rate
   - g. Teaching effectiveness
   - h. Human intelligence
   - i. Air quality
   - j. Wealth of a nation

2. How is each of the above typically quantified?

3. What sorts of events and things do you think primitive humans felt a need to quantify? Make a list. How do you think primitive societies kept track of the values of those quantities?

4. Name some attributes of objects (besides those in number 1 above) that are not quantifiable or that are hard to quantify.

5. Name some quantities for which units of measure have been only recently developed.

Take-Away Message...In this introductory section you have learned to identify quantities and their values and to distinguish between the two. This understanding is required before you can begin the quantitative analysis in the next section.

**1.2 Quantitative Analysis**

In this section you will use what you have learned about quantities in Section 1.1 to analyze problem situations in terms of their quantitative structure. Such analyses are essential to being skillful at solving mathematical problems.

**Definition:** For the purposes of this course, to understand a problem situation means to understand the quantities embedded in the situation and how they are related to one another.

It is this understanding that “drives” the solution to the problem. Without such understanding the only recourse a person has is to guess at the
calculations needing to be performed. It is important that you work through this section with care and attention. Analyzing problem situations quantitatively is central to the remainder of this course and other courses that are part of your preparation to teach elementary school mathematics.

**Activity: The Hot Dog Problem**

| Albert ate 2 \( \frac{3}{4} \) hot dogs and Reba ate 1 \( \frac{1}{2} \) hot dogs. |
| What part of the hot dogs did Albert eat? |

This fairly simple problem is used here to illustrate the process of using a quantitative analysis to solve a problem. We first analyze the situation in terms of the quantities it involves and how those quantities are related to one another: its **quantitative structure**. To do so productively, it is extremely important to be specific about what the quantities are. For example, it is not sufficient to say that “hot dogs eaten” is a quantity in this problem situation. If you indicated “hot dogs eaten” as one of the quantities and specified no more, then someone could ask: Which hot dogs? The ones Albert ate? The ones Reba ate? The total number of hot dog eaten?

**Continuation of Activity...**

*To understand the quantitative structure of this problem situation, we can do the following.*

*First, name as many quantities as you can that are involved in this situation. Be aware that some quantities may not be explicitly stated although they are essential to the situation. Also, just because the value of a quantity is not known or is not given, this does not mean that the quantity is not part of the situation’s quantitative structure.*

*Second, for each quantity, if the value is given write it on the appropriate blank. If the value is not given, indicate that the value is unknown, and write the unit you would use to measure it. You may need more space than provided here. (Reminder: There should be no numbers appearing in the Quantity Column.) Compare your list with others.*

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td></td>
</tr>
<tr>
<td>b.</td>
<td></td>
</tr>
<tr>
<td>c.</td>
<td>etc.</td>
</tr>
</tbody>
</table>
Then, make a drawing that illustrates this problem. Here, the outer rectangle indicates the total number of hot dogs eaten, and the parts show the portions eaten by each person. (Note that drawings such as this one need not be drawn to scale. Here, the number of hot dogs eaten by Albert is shown only to be more than the hot dogs eaten by Reba.)

![Diagram of hot dogs eaten by Reba: 1 1/2 and Hot dogs eaten by Albert: 2 3/4.]

Finally, use the drawing to solve the problem. The total number of hot dogs eaten is \(2 \frac{3}{4} + 1 \frac{1}{2}\) or \(4 \frac{1}{4}\). What part of the total was eaten by Albert? \(\frac{2 \frac{3}{4}}{4 \frac{1}{4}}\) hot dogs, which simplifies to \(\frac{11}{17}\) of the hot dogs. This is about a third of the hotdogs, a more reasonable answer for this question.

The purpose here is to illustrate how one must analyze a situation in terms of the quantities and the relationships present in the situation in order to gain an understanding that can lead to a meaningful solution. Such an activity is called a quantitative analysis. On problems such as this one, you may find that you don’t really need to think through all four steps. However, more difficult problems, such as in Activities 2 and 3, are more easily solved by undertaking a careful quantitative analysis. You will improve your skill at solving problems when you engage in such analyses because you will come to a better understanding of the problem.

**Activity. Sisters and Brothers**

Try this problem by undertaking a quantitative analysis before you read the solution. Then compare your solution to the one shown here.

Two women, Sister A and Sister B, each had a brother, Brother A and Brother B, respectively. The two women argued about which one stood taller over her brother. It turned out that Sister A won the argument by a 17 centimeter difference. Sister A was 186 cm tall. Brother A was 87 cm tall. Sister B was 193 cm tall. How tall was Brother B?
First, name the quantities involved here.

a. The height of Sister A  
b. The height of Sister B.  
c. The height of Brother A  
d. The height of Brother B  
e. The difference between Sister A’s and Brother A’s heights  
f. The difference between Sister B’s and Brother B’s heights  
g. The difference between the differences in the heights of the sister and brother pairs

Next, identify the values of the quantities.

a. The height of Sister A  186 cm  
b. The height of Sister B  193 cm  
c. The height of Brother A  87 cm  
d. The height of Brother B  not known  
e. The difference between Sister A’s and Brother A’s heights  
   not known  
f. The difference between Sister B’s and Brother B’s heights  
   not known  
g. The difference between the differences in the heights of the sister and brother pairs  17 cm  
(The last quantity, but it will be crucial to solving the problem.)

Now draw a picture involving these quantities. Here is one possibility.

![Diagram of heights]

Need to relate the two differences  
(A’s is 17 cm more than B’s):

Perhaps this additional drawing would help clarify the problem: Next to each quantity, place its value if known, and use these values known to find values unknown.

Sister A (186)  
Brother A (87)  
Sister B (193)  
Brother B ????  

Sister A Difference A (99)  
Brother A Difference of differences (17)  
Sister B Difference B ???  
Brother B ????
Finally, use the drawing and/or diagram to solve the problem. Notice that in this case the solution is shown in the first drawing, and can easily be found in the second drawing. But note that the question asked was Brother B’s height. According to the drawings, it is 111 cm.

Often students begin a problem such as this one by asking themselves “What operations do I need to perform, with which numbers, and in what order?” Instead you would be much better off asking yourself questions such as these:

**Quantitative Analysis Questions**

- What do I know about this situation?
- What quantities are involved here? Which ones are critical?
- Are there any quantities that are related to others? If so, how are they related?
- Which quantities do I know the value of?
- Which quantities do I not know the value of? Are these related to other quantities in the situation? Can these relationships enable me to find any unknown values?
- Would drawing a diagram or enacting the situation help me to answer any of these other questions?

And so on.

The point is that you need to be specific when doing a quantitative analysis of a given situation. How specific? It is difficult to say in general, because it will depend on the situation. Use your common sense in analyzing the situation. When you first read a problem, avoid trying to think about numbers and the operations of addition, subtraction, multiplication and division. Instead, start out by posing the questions such as those listed above and try to answer them. Once you’ve done that, you will have a better understanding of the problem and thus you will be well on your way to solving it. Keep in mind that understanding the problem is the most difficult aspect of solving it. Once you understand the problem, *what to do to solve it* usually follows quite easily.

**Activity: Down the Drain**

*Here is another problem situation on which to practice analyzing quantitatively.*
Water is flowing into an empty tub at 4.5 gallons per minute. After 4 minutes, a drain in the tub is opened and the water begins to flow out at 6.3 gallons per minute.

a. Will the tub ever fill up completely?
b. Will it ever empty completely?
c. What if the in-faucet is turned off after 4 minutes?
d. What if the rates of flow are reversed?
e. What assumptions do we have to make in order to answer these questions?

When you carry out a quantitative analysis of a situation, the questions you ask yourself should be guided by your common sense. A sense-making approach to understanding a situation and then solving a problem is much more productive than trying to decide right away which computations, formulas, or mathematical techniques you need in order to solve the problem at hand. Using common sense may lead you to make some sort of a diagram. Never be embarrassed to use a diagram. Such diagrams often enhance your understanding of the situation, because they help you to think more explicitly about the quantities that are involved. Deciding what operations you need to perform often follows naturally from a good understanding of the situation, the quantities in it, and how those quantities are related to one another.

Take-Away Message. . . By now you should be able to determine the quantities and their relationships within a given problem situation, and you should be able to use this information, together with diagrams when needed, to solve problems. These steps are often useful: (1) List the quantities that are essential to the problem. (2) List known values for these quantities. (3) Determine the relationships involved, which is usually more easily done with a drawing. (4) Use the knowledge of these relationships to solve the problem. This approach may not seem very easy at first, but it becomes a powerful tool for understanding problem situations. This type of quantitative analysis can also be used with algebra story problems, and thus, when used with elementary school students, prepares them for algebra.
Learning Exercises for Section 1.2

1. Some problems are simple enough that the quantitative structure is obvious, particularly after a drawing is made. The following problems are from a 5th grade textbook. Make a drawing for each and provide the answer to the problem.

a. The highest elevation in North America is Mt. McKinley, Alaska, which is 20,320 feet above sea level. The lowest elevation in North America is Death Valley, California, which is 282 feet below sea level. What is the change in elevation from the top of Mt. McKinley to Death Valley?

b. The most valuable violin in the world is the Kreutzer, created in Italy in 1727. It was sold at auction for $1,516,000 in England in 1998. How old was the violin when it was sold?

c. Two sculptures are similar. The height of one sculpture is four times the height of the other sculpture. The smaller sculpture is 2.5 feet tall. How tall is the larger sculpture?

d. Aiko had $20 to buy candles. She returned 2 candles for which she had paid $4.75 each. Then she bought 3 candles for $3.50 each and 1 candle for $5.00. How much money did Aiko have then?

e. In Ted’s class, students were asked to name their favorite sport. Football was the response of \( \frac{1}{8} \) of them. If 3 students said football, how many students are in Ted’s class?

f. The first year of a dog’s life equals 15 “human years.” The second year equals 10 human years. Every year thereafter equals 3 human years. Use this formula to find a 6-year-old dog’s age in human years.

2. These problems are from a 6th grade textbook\(^{ii} \) from another series. This time, undertake a full quantitative analysis to solve each of the problems.

a. At Loud Sounds Music Warehouse, CDs are regularly priced at $9.95 and tapes are regularly priced at $6.95. Every day this month the store is offering a 10% discount on all CDs and tapes. Joshua
and Jeremy go to Loud Sounds to buy a tape and a CD. They do not have much money, so they have pooled their funds. When they get to the store, they find that there is another discount plan just for that day—if they buy three or more items, they can save 20% (instead of 10%) on each item. If they buy a CD and a tape, how much money will they spend after the store adds a 6% sales tax on the discounted prices?

b. Kelly wants to fence in a rectangular space in her yard, 9 meters by 7.5 meter. The salesperson at the supply store recommends that she put up posts every 1 ½ meters. The posts cost $2.19 each. Kelly will also need to buy wire mesh to string between the posts. The wire mesh is sold by the meter from large rolls and costs $5.98 a meter. A gate to fit in one of the spaces between the posts costs $25.89. Seven staples are needed to attach the wire mesh to each post. Staples come in boxes of 50, and each box costs $3.99. How much will the materials cost before sales tax?

3. **All Aboard!** Amtrak trains provide efficient, non-stop transportation between Los Angeles and San Diego. Train A leaves Los Angeles headed towards San Diego at the same time that Train B leaves San Diego headed for Los Angeles, traveling on parallel tracks. Train A travels at a constant speed of 84 miles per hour. Train B travels at a constant speed of 92 miles per hour. The two stations are 132 miles apart. How long after they leave their respective stations do the trains meet?

4. My brother and I walk the same route to school every day. My brother takes 40 minutes to get to school and I take 30 minutes. Today, my brother left 8 minutes before I did.

   a. How long will it take me to catch up with him?

   b. Part of someone’s work on this problem included \(\frac{1}{30} - \frac{1}{40}\). What quantities do the two fractions \(\frac{1}{30} - \frac{1}{40}\) represent?

   c. Suppose the time difference is 5 minutes instead of 8 minutes. Now how long does it take for me to catch up with him?

5. At one point in a Girl Scout cookie sales drive,

   Region C had sold 1500 boxes of cookies, and
Region D had sold 1200 boxes of cookies.

If Region D tries harder, they can sell 50 more boxes of cookies every day than Region C can.

a. How many days will it take for Region D to catch up?

b. If sales are stopped after eight more days, can you tell how many total boxes each Region sold? Explain.

6. The last part of the triathlon is a 10K (10 kilometers, or 10 000 meters) run. When runner Aña starts this last running part, she is 600 meters behind runner Bea. But Aña can run faster than Bea: Aña can run (on average) 225 meters each minute, and Bea can run (on average) 200 meters each minute. Who wins, Aña or Bea? If Aña wins, when does she catch up with Bea? If Bea wins, how far behind is Aña when Bea finishes?

7. Research on how students solve word problems contained the following incident.iv Dana, a seventh grader in a gifted program in mathematics, was asked to work the following problem:

A carpenter has a board 200 inches long and 12 inches wide. He makes 4 identical shelves and still has a piece of board 36 inches long left over. How long is each shelf?

Dana tried to solve the problem as follows: She added 36 and 4, then scratched it out, and wrote 200 × 12, but she thought that was too large so she scratched that out. Then she tried 2400 − 36 which was also too large and discarded it. Then she calculated 4 × 36 and subtracted that from 200, getting 56. She then subtracted 12, and got 44.

Dana used a weak strategy called “Try all operations and choose.” She obviously did not know what to do with this problem, although she was very good at solving one-step problems.

Do a quantitative analysis of this problem situation, and use it to make sense of the problem in a way that Dana could not. Use your analysis to solve the problem.

8. These problems, and problem 9, are from a Soviet Grade 3 textbookv. Solve the problems and compare their conditions and solutions:
(1) Two pedestrians left two villages simultaneously and walked towards each other, meeting after 3 hours. The first pedestrian walked 4 km in an hour, and the second 6 km. Find the distance between their villages:

(2) Two pedestrians left two villages 27 km apart simultaneously and walked towards each other. The first one walked 4 km per hour, and the second 6 km per hour. After how many hours did the pedestrians meet?

(3) Two pedestrians left two villages 27 km apart simultaneously and walked towards each other, meeting after 3 hours. The first pedestrian walked at a speed of 4 km per hour. At what speed did the second pedestrian walk?

9. Two trains simultaneously left Moscow and Sverdlovsk, and traveled towards each other. The first traveled at 48 km per hour, and the second at 54 km per hour. How far apart were the two trains 12 hours after departure if it is 1,822 km from Moscow to Sverdlovsk?

1.3 Values of Quantities

The value of a quantity may involve very large or very small numbers. Furthermore, since the value is determined by counting or other ways of
measuring, it can involve any type of number—whole numbers, fractions, decimals.

_Think About 1…_ Consider the following quantities. Which ones would you expect to have large values? Small values?

a. The distance between two stars
b. The diameter of a snowflake
c. The weight of an aircraft carrier
d. The national deficit
e. The thickness of a sheet of paper

In one book the height of the arch in St. Louis is reported to be 630 feet. Another book states that the height is 192 meters.

_Think About 2…_ What determines the magnitude of the number that denotes the value of a given quantity? Can we measure the speed of a car in miles per day? Miles per year? Miles per century? Is it convenient to do so? Explain.

Discussion: Units of Measure

1. What determines the appropriateness of the unit chosen to express the value of a quantity?
2. For each quantity listed above (a-e), name a unit of measure that would be appropriate to measure the quantity.
3. Explain how you would determine the thickness of a sheet of paper.
4. What determines the “precision” of the value of a quantity?

Unless one has some appreciation and understanding of the magnitude of large numbers, it is impossible to make judgments about such matters as the impact of a promise of five million dollars in relief funds after a catastrophic flood or earthquake, or the level of danger of traveling in a country where there have been three known terrorist attacks in the last year, or the personal consequences of the huge national debt, or the meaning of costly military mistakes. For example, people are shocked to hear that the Pentagon spent $38 for each simple pair of pliers bought from a certain defense contractor, but pay little attention to the cost of building the Stealth fighter, or the cost of losing a jet fighter during testing.
Activity: Jet Fighter Crashes

In the 1990s, a West Coast newspaper carried a brief article saying that a $50 million jet fighter crashed into the ocean off the California coast. How many students could go to your university tuition free for one year with $50 million?

Discussion: What Is Worth A Trillion Dollars?

Suppose you hear a politician say "A billion dollars, a trillion dollars, I don't care what it costs, we have got to solve the AIDS problem in this country." Would you agree? Is a billion dollars too much to spend on a national health crisis? A trillion dollars? How do the numbers one billion and one trillion compare?

Reminder: Values of quantities, like 16 tons or $64, involve units of measure—ton, dollar—as well as numbers.

Think About... Name several units of measure that you know and use. What quantities are each used to measure? Where do units come from?

Units can be arbitrary. Primary teachers have their students measure lengths and distances with pencils or shoe lengths, or weights with plastic cubes. The intent of these activities is to give the children experience with the measuring process, so that later measurements will make sense. As you know, there are different systems of standard units, like the English or “ordinary” system (inches, pounds, etc.) and the metric system (meters, kilograms, etc.), or more formally known as SI (from Le Système International d'Unités).

Virtually the rest of the world uses the metric system to denote values of quantities, so many are surprised that the United States, a large industrial nation, has clung to the English system so long. Although the general public has not responded favorably to governmental efforts to mandate the metric system, international trade efforts are having the effect of forcing us to be knowledgeable about, and to use, the metric system. Some of our
largest industries have been the first to convert to the metric system from the English system.

**Discussion: Standard Units**

*Why are standard units desirable? For what purposes are they necessary? Why has the public resisted adopting wholeheartedly the metric system?*

Scientists have long worked almost exclusively in metric units. As a result, you may have been introduced to metric units in your high school or college science classes. Part of the reason for this is that the rest of the world uses the metric system because it is a sensible system. A basic metric unit is carefully defined (for the sake of permanence and later reproducibility). Larger units and smaller sub-units are related to each other in a consistent fashion, so it is easy to work within the system. (In contrast, the English system unit, foot, might have been the length of a now-long-dead king’s foot, and it is related to other length units in an inconsistent manner: 1 foot = 12 inches; 1 yard = 3 feet; 1 rod = 16.5 feet; 1 mile = 5280 feet; 1 furlong = \( \frac{1}{8} \) mile; 1 fathom = 6 feet. Quick!—how many rods are in a mile? A comparable question in the metric system is just a matter of adjusting a decimal point.

To get a better idea of how the metric system works, let’s consider something that we frequently measure in metric units—length. Length is a quality of most objects. We measure the lengths of boards and pieces of rope or wire. We also measure the heights of children; height and length refer to the same quality but the different words are used in different contexts.

*Think About...* What are some other words that refer to the same quality as do “length” and “height”?

The basic SI unit for length (or its synonyms) is the *meter*. (The official SI spelling is *metre*. You occasionally see “metre” in U.S. books.) The meter is too long to show with a line segment here, but two sub-units fit easily, and illustrate a key feature of the metric system: units smaller and larger than the basic unit are multiples or sub-multiples of powers of 10.
Notes

0.1 meter ——

0.01 meter ——

Furthermore, these sub-units have names—decimeter, centimeter—which are formed by putting a prefix on the word for the basic unit. The prefix “deci-” means one-tenth, so “decimeter” means 0.1 meter. Similarly, “centi-” means one one-hundredth, so “centimeter” means 0.01 meter. You have probably heard “kilometer;” the prefix “kilo-” means 1000, so “kilometer” means 1000 meters. On reversing one's thinking, so to speak, there are 10 decimeters in 1 meter, there are 100 centimeters in 1 meter, and 1 meter is 0.001 kilometer.

Another feature of SI is that there are symbols for the basic units—m for meter—and for the prefixes—d for deci-, c for centi-, and so on—so a length measurement can be reported quite concisely: 18 cm, 2.3 dm. The symbols, cm and dm, do not have periods after them, nor do the abbreviations in the English system: ft, mi, etc, except that for inch (in.).

If you are new to the metric system, your first job will be to familiarize yourself with the prefixes so you can apply them to the basic units for other qualities. The table below shows some of the other metric prefixes. For example, “kilo-” means 1000, so “1 kilometer” means 1000 meters; the symbol k for kilo- and the symbol m for meter give km for kilometer.

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Symbol</th>
<th>Meaning of Prefix</th>
<th>Applied to length</th>
</tr>
</thead>
<tbody>
<tr>
<td>kilo-</td>
<td>k</td>
<td>1 000 or $10^3$</td>
<td>km</td>
</tr>
<tr>
<td>hecto-</td>
<td>h</td>
<td>100 or $10^2$</td>
<td>hm</td>
</tr>
<tr>
<td>deka-</td>
<td>da</td>
<td>10 or $10^1$</td>
<td>dam</td>
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<tr>
<td>no prefix</td>
<td></td>
<td>1 or $10^0$</td>
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<td>deci-</td>
<td>d</td>
<td>0.1 or $10^{-1}$</td>
<td>dm</td>
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<tr>
<td>centi-</td>
<td>c</td>
<td>0.01 or $10^{-2}$</td>
<td>cm</td>
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<tr>
<td>milli-</td>
<td>m</td>
<td>0.001 or $10^{-3}$</td>
<td>mm</td>
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</tbody>
</table>

Activity: It’s All in the Unit

1. Measure the width of your desk or table, in decimeters. Express that length in centimeters, millimeters, meters, and kilometers.
2. Measure the width of your desk or table, in feet. Express that length in inches, yards, and miles.
3. In which system are conversions easier? Explain why.
Take-Away Message: The value of a quantity is expressed using a number and a unit of measure. Commerce depends upon having agreed sets of measures. Standard units accomplish this. The common system of measurement in the United States is the called the English system of measurement. We use the metric system, another measurement system, for science and for international trade. Most countries of the world use only the metric system. The metric system is based on powers of ten and on common prefixes, making the system an easier one to use.

**Learning Exercises for Section 1.3**

1. Name an appropriate unit for measuring:
   a. the amount of milk that a mug will hold.
   b. the height of the Empire State Building.
   c. the distance between San Francisco and New York.
   d. the capacity of the gas tank in your car.
   e. the safety capacity of an elevator.
   f. the amount of rainfall in one year

2. a. What does it mean to say that “a car gets good mileage?”
   
   b. What unit is used to express “gas mileage?”
   
   c. How could you determine the mileage you get from your car?
   
   d. Do you always get the same mileage from your car? List some factors that influence how much mileage you get.
   
   e. How would you measure “gas consumption?” Is gas consumption related to mileage? If so, how?

3. Explain how rainfall is quantified. You may need to use resources such as an encyclopedia.

4. a. Calculate an approximation of the amount of time you have spent sleeping since you were born. Explain your calculations. Express your approximation in hours, in days, in years.

   b. What part of your life have you spent sleeping?
c. On the average how many hours do you sleep each day? What fractional part of the day is this?

d. How does your answer in b compare to your answer in c?

5. Name an item that can be used to estimate the following metric units:
   a. a centimeter  
   b. a gram   
   c. a liter   
   d. a meter   
   e. a kilometer   
   f. a kilogram

6. There are some conversions from English to metric units that are commonly used, particularly for inch, mile, and quart. What are they?

1.4 Issues for Learning

In a study\textsuperscript{vi} of how children learn story problems students were found to use seven different strategies. The first six strategies are all based on something other than understanding the problem.

1. Find the numbers in the problem and just do something to them, usually add since that is the easiest operation.

2. Guess at the operation to be used, perhaps based on what had been most recently studied.

3. Let the numbers “tell” you what to do. One student said, “if it’s like 78 and maybe 54, then I’d probably either add or multiply. But if the numbers are 78 and 3, it looks like a division because of the size of the numbers.”

4. Try all the operations and then choose the most reasonable answer. This strategy often worked for one step problems, but would not for two step problems.

5. Look for “key” words to decide what operation to use. For example, “all together” means to add. This strategy works sometimes, but not all the time. Also, word like “of” and “is” would signify multiplication, and equals, but some students confuse the two.

6. Narrow the choices, based on expected size of the answer. For example, when a student used division on a problem involving
reduction in a photocopy machine, he said he did because "it's reducing something, and that means taking it away or dividing it."

7. Choose an operation based on understanding the problem. Often students would make a drawing when they used this strategy. Unfortunately, though, few of the children (6th and 8th graders, of average or above average ability in mathematics) used this strategy.

Only the last strategy was considered a mature strategy. These children understood the problem because they had undertaken a quantitative analysis of the problem, even though not so formally as introduced in this chapter.

Making drawings plays an essential role in coming to understand a problem. Here is an excerpt from one interview with one of the children in this study:

Emmy: I just pictured the post, how deep the water was...Sometimes I picture the objects in my mind that I'm working with, if it's a hard problem...

Interviewer: Does that help?

Emmy: Yeh, it helps. That's just one way of, kind of cheating, I guess you'd say.

Unfortunately, too many students have come to believe that making drawings is cheating, or is juvenile.

In another study vii, two researchers compared how drawings are used in the U.S. textbooks and in Japanese textbooks. They found that many elementary school students in the U.S. are not encouraged to make drawings that will help them understand a story problem. However, even flipping through the pages of Japanese textbooks shows that drawings are used throughout. Teachers say to students, "If you can draw a picture, you can solve the problem."

We hope that the work in this chapter and in future ones will help you use the 7th, more mature strategy if you do not already do so, and to
understand how children can be taught to reason at this level. Analyzing situations quantitatively, which includes making drawings, can help one understand the problems being solved. “Quantitative reasoning is more than reasoning about numbers, and it is more than skilled calculating. It is about making sense of the situation to which we apply numbers and calculations.” viii

1.5 Check Yourself

In this first chapter, you have learned about the role quantities play in our lives and the ways we express quantities and their values. You have learned about dealing with problem situations by analyzing the problem in terms of its quantities and their relationships to one another. This quantitative analysis helps solve a problem in a meaningful, sense-making way. The same kind of analysis can be applied to arithmetic problems and to algebra problems.

You also learned how quantities are measured in terms of units, and how to express values of quantities in standard units, including metric units.

You should be able to work problems like those assigned and to do the following.

1. Identify the quantities addressed by questions such as: How much damage did the flooding cause?

2. Discuss the importance of quantification on the advance of culture.

3. Name attributes of objects that cannot be quantified.

4. Distinguish between a quantity and its value.

5. Given a problem situation, undertake a quantitative analysis of the problem and use that analysis to solve the problem.

6. Determine appropriate units to measure quantities.

7. Discuss reasons why the metric system is used for measurement in most countries.

8. Discuss some incorrect ways that children solve story problems.
References for Chapter 1:


Chapter 2  Numeration Systems

Contrary to what you may believe, there are many ways of expressing numbers. Some of these ways are cultural and historical. Others are different ways of thinking about what the digits of our conventional number system mean. For example, you probably think of 23 as meaning two tens and three ones. We think this way because we use a base ten system of counting. (How many fingers do you have?) But why not base five (using only five fingers)? What would 23 mean then? You are about to find out.

2.1 Ways of Expressing Values of Quantities

The need to quantify and express the values of quantities led humans to invent numeration systems. Throughout history, people have found ways to express values of quantities they measured in a variety of ways. A variety of words and special symbols, called numerals, have been used to communicate number ideas. How one expresses numbers using these special symbols makes up a numeration system. Our Hindu-Arabic system uses ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Virtually all present-day societies use the Hindu-Arabic numeration system. With the help of decimal points, fraction bars, and marks like square root signs, these ten digits allow us to express almost any number and therefore the value of almost any quantity. What are some exceptions?

Think About...Why are numerals used so much? What are advantages of these special symbols over using just words to express numbers? What are some exceptions to representing numbers with digits?

Activity: You Mean People Didn’t Always Count the Way We Do?
A glimpse of the richness of the history of numeration systems lies in looking at the variety of ways in which the number twelve has been expressed. In the different representations shown, see if you can deduce what each individual mark represents. Each representation expresses this many:  ⦁⦁⦁⦁⦁⦁⦁⦁⦁⦁⦁⦁
Think About... How would ten have been written in each of these earlier numeration systems?

Some ancient cultures did not need many number words. For example, they may have needed words only for “one,” “two,” and “many.” When larger quantities were encountered, they could be expressed by some sort of matching with pebbles or sticks or parts of the body, but without the use of any distinct word or phrase for the number involved. For example, in a recently-discovered culture in Papua New Guinea, the same word “doro” was used for 2, 3, 4, 19, 20, and 21. But by pointing also to different parts of the hands, arms, and face when counting and saying “doro,” these people could tell which number is intended by the word. This method of pointing allows the Papua New Guineans to express numbers up through 22 easily.¹ It was only when this culture came into contact with the outside world and began trading with other cultures that they needed to find ways of expressing larger numbers.

Think About... Why do you think we use ten digits in our number system? Would it make sense to use twenty? Why or why not?

Discussion: Changing Complexity of Quantities Over Time

What quantities, and therefore what number words, would you expect a cave-man to have found useful? (Assume that the caveman had a sufficiently sophisticated language.) A person in a primitive agricultural society? A pioneer? An ordinary citizen living today? A person on Wall Street? An astronomer? A subatomic physicist?
Take-Away Message…Mathematical symbols have changed over the years, and they may change in the future. Symbols used for numbers depend upon our need to determine the value of the quantities with which we work.

**Learning Exercises for Section 2.1**

1. Based on what you have seen of the old counting systems such as Greek, Chinese, Roman, Babylonian, Mayan, and Aztec, which systems make the most sense to you? Explain.

2. Symbols for five and for ten often have had special prominence in geographically and chronologically remote systems. Why?

3. Numbers can be expressed in a fascinating variety of ways. Different languages, of course, use different words and different symbols to represent numbers. Some counting words are given below.

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<tbody>
<tr>
<td>English</td>
<td>Spanish</td>
<td>German</td>
<td>French</td>
<td>Japanese</td>
<td>Swahili</td>
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<td>zero</td>
<td>cero</td>
<td>null</td>
<td>zero</td>
<td>zero</td>
<td>sifuri</td>
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<tr>
<td>one</td>
<td>uno</td>
<td>eins</td>
<td>un</td>
<td>ichi</td>
<td>moja</td>
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<tr>
<td>two</td>
<td>dos</td>
<td>zwei</td>
<td>deux</td>
<td>ni</td>
<td>mbili</td>
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<tr>
<td>three</td>
<td>tres</td>
<td>drei</td>
<td>trois</td>
<td>san</td>
<td>tatu</td>
</tr>
<tr>
<td>four</td>
<td>cuatro</td>
<td>vier</td>
<td>quatre</td>
<td>shi</td>
<td>nne</td>
</tr>
<tr>
<td>five</td>
<td>cinco</td>
<td>fünf</td>
<td>cinq</td>
<td>go</td>
<td>tano</td>
</tr>
<tr>
<td>six</td>
<td>seis</td>
<td>sechs</td>
<td>six</td>
<td>roku</td>
<td>sita</td>
</tr>
<tr>
<td>seven</td>
<td>siete</td>
<td>sieben</td>
<td>sept</td>
<td>shichi</td>
<td>saba</td>
</tr>
<tr>
<td>eight</td>
<td>ocho</td>
<td>acht</td>
<td>huit</td>
<td>hachi</td>
<td>nane</td>
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<tr>
<td>nine</td>
<td>nueve</td>
<td>neun</td>
<td>neuf</td>
<td>kyu</td>
<td>tisa</td>
</tr>
<tr>
<td>ten</td>
<td>diez</td>
<td>zehn</td>
<td>dix</td>
<td>ju</td>
<td>kumi</td>
</tr>
</tbody>
</table>

Which two sets of these counting words most resemble one another?

Why do you think that is true? Do you know these numbers in yet another language?
4. Roman numerals have survived to a degree, as in motion picture film credits and on cornerstones. Here are the symbols: I = one, V = five, X = ten, L = fifty, C = one hundred, D = five hundred, and M = one thousand. For example, CLXI is 100 + 50 + 10 + 1 = 161.

a. MMCXIII      b. CLXXXV  c. MDVII

5. How would each of the following be written in Roman numerals? For example, one thousand one hundred thirty would be MCXXX.

a. two thousand sixty-six  b. seventy-eight  c. six hundred five

6. Other systems we have seen all involve addition of the values of the symbols. Roman numerals use a subtractive principle as well; when a symbol for a smaller value comes before the symbol for a larger value, the former value is subtracted from the latter. For example, IV means 5 - 1 = 4, or four; XC means 100 - 10 = 90; and CD = 500 - 100 = 400. Note that no symbol appears more than three times together, because with four symbols we would use this subtractive property. What number does each of these represent?

a. CMIII      b. XLIX  c. CDIX

7. Even within the same language, there are often several words for a given number idea. For example, both “two shoes” and “a pair of shoes,” refer to the same quantity. What are some other words for the idea of two-ness?

2.2 Place Value

What does each 2 in 22,222 mean? The different 2s represent different values because our Hindu-Arabic numeration system is a place-value system. This system depends upon powers of ten to tell us the meaning of each digit. Once this system is understood, arithmetic operations are much easier to learn. Understanding of place value is a fundamental idea underlying elementary school mathematics.

But first, what does it mean to have a place-value system?

Definition: In a place-value system, the value of a digit in a numeral is determined by its position in the numeral.
Example: In 506.7, the 5 is in the hundreds place, so it represents five hundred. The 0 in 506.7 is in the tens place, so it represents zero tens, or just zero. The 6 is in the ones place, so it represents six ones, or six. And the 7 is in the tenths place, so it represents seven-tenths. The complete 506.7 symbol then represents the sum of those values: five hundred six and seven-tenths.

Notice that we do not say “five hundred, zero tens, six and seven-tenths,” although we could. This is symptomatic of the relatively late appearance, historically, of a symbol for zero. The advantage of having a symbol to say that nothing is there is apparently a difficult idea, but the idea is vital to a place-value system. Would 506.7 mean the same number if we omitted the 0 to get 56.7? The 0 may have evolved from some type of round mark written in clay by the Babylonians to show that there are zero groups of a particular place value needed.

*Think About...* In the Hindu-Arabic place-value system, how many different places (positions) can you name and write numerically? (Don’t forget places to the right of the decimal point.)

Before the use of numerals became widespread, much calculation was done with markers on lines for different place values. The lines could be on paper, or just drawn in sand, with small stones used as markers. (Our word “calculate” comes from the Latin word for “stones.”) One device that no doubt was inspired by these methods of calculating is the **abacus**, which continues to be used in some parts of the world.

![Abacus](image)

A Chinese Abacus

Notice that we have often used words to discuss the numbers instead of the usual numerals. The reason is that the symbol “12” is automatically associated with “twelve” in our minds because of our familiarity with the
usual numeration system. We will find that the numeral “12” could mean five or six, however, in other systems! (If no base is indicated, assume the familiar base ten is indicated.)

In our base-ten numeration system, the whole-number place values result from groups of ten—ten ones, ten tens, ten hundreds, etc. The digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 work fine until we have ten of something. But there is no single digit that means ten. When we have ten ones in our base-ten system, we think of them as one group of ten, without any left-over ones. We also take advantage of place value to write “10,” one ten and zero left-over ones.

In our base-ten system when we have ten ones, we think of them as one group of ten, without any left-over ones, and we take advantage of place value to write “10”, one ten and zero left-over ones. Similarly, two place values are sufficient through nine tens and nine ones, but when we have ten tens, we then use the next place value and write “100.” It is like replacing ten pennies with one dime, or trading ten dimes for a dollar.

Examples: If I want to find the number of ten dollar bills I could get for $365, the answer is not just 6, it is 36.

If I want to know how many dollars bills I could get from $365, the answer is 365.

If I want to know the number of dimes I could get from $365, the answer is 3650.

But if I want to know how many tens are in 365, I could say either 36 or 36.5, depending on the context.

If I have 365 bars of soap and I want to know how many full boxes of 10 I could pack, the answer would be 36.

If I am buying 365 bars of soap priced at $6 per 10 bars, then I would have to pay 36.5 times $6.

With a good understanding of place value, the problems like those in the example can be easily solved without undertaking long division or multiplication by 10 or powers of 10. Children who do not understand place value will often try to solve the problem of how many tens are in
365 by using long division to divide by 10, rather than observing that the answer is obvious from the number.

**Discussion: Money and Place Value**

*Explain your answers to each of the following:*

1. How many ten-dollar bills does the 6 in $657$ represent? The 5?
2. How many tens are in 657?
3. How many one-hundred dollar bills can you get for $53,908$?
4. How many one-hundreds are in 53,908
5. How many pennies can you get for $347$? for $34.70$? for $3.47$?
6. How many ones are in $347$? In $34.70$? in $3.47$

The decimal point indicates that we are beginning to break up the unit one into tenths, hundredths, thousandths, etc. But the number *one*, not the decimal point, is the focal point of this system. So 0.642 is 642 thousandths of *one*. Put another way, 0.6 is six tenths of *one*, while 6 is six *ones*, and 60 is six tens, or 60 *ones*. But just as 0.6 is six tenths of one, 6 is six tenths of 10, 60 is six tenths of one hundred, and so on up the line. Or starting with smaller numbers, 0.006 is six tenths of 0.01, while 0.06 is six tenths of 0.1. Likewise, 6000 is 60 hundreds, 600 is 60 tens, 60 is 60 ones, 6 is 60 tenths, 0.6 is 60 hundredths, 0.06 is 60 thousandths, and so on. While this at first might seem confusing, it becomes less so with practice.

Take-Away Message...Our base 10 place-value numeration system is adequate for expressing all whole numbers and many decimal numbers. The value of each digit in a numeral is determined by the position of the digit in the numeral. Digits in different places have different values. The number 1 serves as the focal point of decimal numbers, not the decimal point. When we state a whole number such as 341, we really mean 341 *ones*.

**Learning Exercises for Section 2.2**

1. a. How many tens are in 357? How many whole tens?

   b. How many hundreds are in 4362? How many whole hundreds?

   c. How many tens are in 4362? How many whole tens?

   d. How many thousands are in 456,654? How many whole thousands?

   e. How many hundreds are in 456,654? How many whole hundreds?
f. How many tens are in 456,654? How many whole tens?

2. In 123.456, the hundredths place is in the third place to the left of the decimal point; is the hundredths place in the third place to the right of the decimal point? In a long numeral like 333331.333333, what separates the number into two parts that match in the way hundreds and hundredths do?

3. a. “For a set of whole numbers, the longest numeral will belong to the largest number.” True or false? Why?
   b. “For a set of decimals, the longest numeral will belong to the largest number.” True or false? Why?

4. Pronounce 3200 in two different ways. Do the two pronunciations have the same value?

5. Write in words the way you would pronounce each:
   a. 407.053  b. 30.04   c. 0.34   d. 200.067   e. 0.276

6. Each of the following represents work of students who did not understand place value. Find the errors made by these students, and explain their reasoning.

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<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>115</td>
<td>55</td>
<td>4</td>
</tr>
<tr>
<td>+ 96</td>
<td>+ 48</td>
<td>- 2</td>
</tr>
<tr>
<td>1010</td>
<td>913</td>
<td>1</td>
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<td></td>
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<td>d</td>
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<td>21</td>
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</tbody>
</table>
7. In base ten, 1635 is exactly _______ ones, is exactly _______ tens, is exactly _______ hundreds, is exactly _______ thousands; it is also exactly _______ tenths, or exactly _______ hundredths.

8. In base ten, 73.5 is exactly _______ ones, is exactly _______ tens, is exactly _______ hundreds, is exactly _______ thousands; it is also exactly _______ tenths, or exactly _______ hundredths.

9. Do you change the value of a whole number by placing zeros to the right of the number? To the left of the number?

2.3 Bases Other Than Ten

Too often children learn to operate on numbers without having a deep understanding of place value, which leads them to make many computational errors. The purpose of this section is to provide experiences with base numeration systems other than ten so you understand the underlying structure of the base ten system of numeration. You are not expected to become fluent in a base other than ten. Rather, you should be able to calculate in different bases to the extent that is needed to understand the role of place value, particularly in calculations.

Think About... We use a base ten system of counting because we have ten fingers. Other cultures have used other bases. For example, some Eskimos were found to count using base five. Why would that be? What other bases might have been used for counting?

Cartoon characters often have three fingers and a thumb on each hand, a total of eight fingers (counting thumbs) instead of ten. Suppose that we live in this cartoon land and instead of having ten digits in our counting system (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) we have only eight digits (0, 1, 2, 3, 4, 5, 6, 7). Using this new counting system we write the number eight as 10, meaning 1 group of eight and 0 ones. Thus, we would write as we count in base eight:

1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, ...
read as: one, two, three, . . . one-zero, one-one, one-two, . . . two zero . .  

**Activity: Place Value in Cartoon Land**  
1. Show the value of each place in base eight by completing this pattern:  
   \[ \cdots \quad 8^5 \quad 8^4 \quad 8^3 \quad 8^2 \quad 8^1 \quad 8^0 \]  
   \( \? \quad \? \quad \? \quad \text{sixty-fours} \quad \text{eights} \quad \text{ones} \)  
2. What would follow 77 in base eight?  
3. What would each digit indicate in the numeral 743 in base eight?  

Notice that the base-eight numeration system has eight digits, 0–7. Writing 6072 in base eight would require the use of the first four places to the left of the decimal point and represents 2 ones \((8^0)\), 7 eights \((8^1)\), 0 sets of eight squared \((8^2)\), and 6 sets of eight cubed \((8^3)\). The digits 6, 0, 7, and 2 would be placed in the Activity pattern in the four places to the left of the decimal point. We call this number “six zero seven two, base eight” and write it as 6072\text{eight}.

**Think About...** If you had 602\text{eight} chairs in an auditorium, how many chairs would you have, counting in base ten?  

**Discussion: Place Value in Base Three**  
What are the place values in a base three system? What are the digits, and how many do we need? (Rather than invent new symbols for digits, let’s use whichever of the standard symbols we need.) Study the chart below. What should be in place of the question marks?

<table>
<thead>
<tr>
<th>Items</th>
<th>Name in base ten</th>
<th>Name in base three</th>
<th>Base three symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>zero</td>
<td>zero</td>
<td>(0\text{ three} )</td>
</tr>
<tr>
<td></td>
<td>one</td>
<td>one</td>
<td>(1\text{ three} )</td>
</tr>
<tr>
<td></td>
<td>two</td>
<td>two</td>
<td>(2\text{ three} )</td>
</tr>
<tr>
<td></td>
<td>three</td>
<td>???</td>
<td>???</td>
</tr>
</tbody>
</table>
Naming three in base three is a key step in understanding base three. Since there are three single boxes above, they will be grouped to make one group of three, and the base three symbol is 10! Notice that in base three “10” does not symbolize ten as we think about ten. In base three, “10” means “one group of three and zero left over.” Since it does not mean ten, we should not pronounce the numeral as “ten.” The recommended pronunciation is “one zero, base three,” saying just the name for each digit and for the base. Notice how this chart differs from the one above.

<table>
<thead>
<tr>
<th>Items</th>
<th>Name in base ten</th>
<th>Name in base three</th>
<th>Base three symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>zero</td>
<td>zero</td>
<td>0 three</td>
</tr>
<tr>
<td>□</td>
<td>one</td>
<td>one</td>
<td>1 three</td>
</tr>
<tr>
<td>□ □</td>
<td>two</td>
<td>two</td>
<td>2 three</td>
</tr>
<tr>
<td>□□□</td>
<td>three</td>
<td>one-zero</td>
<td>10 three</td>
</tr>
<tr>
<td>□□□ □</td>
<td>four</td>
<td>one-one</td>
<td>11 three</td>
</tr>
</tbody>
</table>

If there are four boxes, as in the last line of this table, we can make one group of three, and then there will be one left-over box, so in base three, four is written “11.” Because we have the strong link between the marks “11” and eleven from all of our base ten experience, the notation \(11_{\text{three}}\) is often used for clarity to show that the symbols should be interpreted in base three. Recall that \(11_{\text{three}}\) should be pronounced “one one, base three,” and not as “eleven.”

**Activity: Count in Base Three**

*Continue to draw more boxes and to write base three symbols. What do you write for five boxes? (Now you see why the symbol 12 might mean five.) Six? Seven? Eight? And, at another dramatic point, nine? Did you write “100\(_{\text{three}}\)” for nine? What would 1000\(_{\text{three}}\) mean?*

*Check your counting skills by following along with counting in base four:  1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33, 100, 101, 102, 103, 110, 111, 112, 113, 120, 121, 122, 123, 130, 131, 132, 133, 200 . . .  
What does 1000\(_{\text{four}}\) mean?*
Discussion: Working With Different Bases

1. What are the place values in base five? What digits are needed? How would thirty-eight (in base ten) be expressed in base five? Record the first fifteen counting numbers in base five: 1, 2, ...

2. What are the place values in a base b place-value system? What digits are needed?

3. What are the place values two place-value system. How would eighteen (in base ten) be written in base two? The inner workings of computers use base two; do you see any reason for this fact?

4. Perhaps surprisingly, there is a Duodecimal Society, which promotes the adoption of a base twelve numeration system. What are the place values in a base twelve system? What new digits would have to be invented?

With several numeration systems possible, there can be many “translations” among the symbols. For example, given a base ten numeral (or the usual word), find the base six (or four or twelve) numeral for the same number, and vice versa, given a numeral in some other base, find its base ten numeral (or the usual word). In each case, the key is knowing, and probably writing down, the place values in the unfamiliar system. (Recall the any number to the 0 power is 1. Example: \(5^0 = 1\).)

Example: Changing from a non-ten base to base ten: What does 2103\(_{\text{four}}\) represent in base ten?

1. 2103\(_{\text{four}}\) has four digits. The first four place values in base four are written here, and the given digits put in there places:

\[
\begin{align*}
\text{2} & & \text{1} & & \text{0} & & \text{3} \\
\text{of four sixteens,} & & \text{of four fours, or} & & \text{of four ones, or} & & \text{ones, or} \\
\text{or sixty-four, or} & & \text{sixteen, or} & & \text{four, or} & & \text{4}^0 \\
& & \text{4}^3 & & \text{4}^2 & & \text{4}^1
\end{align*}
\]

2. What does the 2 tell us? The 2 stands for two of \(4^3\) which is \(2 \times 64 = 128\) in base ten.

3. What does the 1 tell us? The 1 stands for one \(4^2\) which is \(1 \times 16 = 16\) in base ten.

4. What does the 0 tell us? The 0 stands for zero of \(4^1\) which is \(0 \times 4 = 0\) in base ten.

5. What does the 3 tell us? The 3 indicates \(4^0\) is used three times, \(3 \times 1 = 3\) in base ten.
6. Thus $2103_{\text{four}} = (128 + 16 + 0 + 3)_{\text{ten}} = 147_{\text{ten}}$
that is, $2103_{\text{four}} = 147_{\text{ten}}$.

Example: Suppose instead we want to change a number written in
base ten, say 236, to a number written in another base, say base
five. We know that the places in base five are the following:

\[
\begin{array}{cccc}
\cdots & \text{one-hundred} & \text{twenty-fives (5}^2) & \text{fives (5}^1) & \text{ones (5}^0) \\
\text{twenty fives (5}^3) & & & & \\
\end{array}
\]

Solution: (You may find these steps easier to follow by dropping the
ten subscript for now, for numbers in base ten.)

1. Look for the highest power of 5 in the base ten number; here it is $5^3$
because $5^4$ is $625_{\text{ten}}$ and $625_{\text{ten}}$ is larger than $236_{\text{ten}}$. Are there any
$5^3$s in $236_{\text{ten}}$? Yes, just one $5^3$ because $5^3 = 125$, and there is only
one 125 in 236. Place a 1 in the first place above to indicate one $5^3$.
Now you have “used up” 125, so subtract: $236_{\text{ten}} - 125_{\text{ten}} =
111_{\text{ten}}$.

2. The next place value of five is $5^2$. Are there any twenty-fives in
$111_{\text{ten}}$? There are 4, so place a 4 above $5^2$. Now four twenty-fives,
or 100, has been “used”, and $111_{\text{ten}} - 100_{\text{ten}} = 11_{\text{ten}}$.

3. The next place value is $5^1$ which is 5. How many fives are in $11_{\text{ten}}$?
It has two fives, so place 2 above $5^1$. There is 1 one left, so place a
1 above $5^0$. Thus $236_{\text{ten}} = 1421_{\text{five}}$.

Working with different bases can be easier when one can physically move
pieces that represent different values in a base system. Often, after doing
physical manipulation, one can mentally picture the manipulation and
work without physical objects. Multibase blocks are manipulatives that
have proven to be extremely useful in coming to understand any base
system, but primarily base ten in elementary school. Multibase blocks are
wooden or plastic blocks that can be used to demonstrate operations in
different bases. For base ten, a centimeter cube can be used to represent a
unit; a long block one centimeter by one centimeter by ten centimeters
(often marked in ones) would then represent ten; ten longs together form a
flat that is one cm by ten cm by ten cm and that represents one hundred,
ten flats form a ten cm by ten cm by ten cm cube that represents thousands. If the long is used for the unit, then the small cube would represent one-tenth, the flat would represent ten, and so on. The multibase blocks can be used to strengthen place value understanding.

If the multibase blocks are not available, then they can easily be sketched as shown below:

The materials are often called “small cube, long, flat, big cube.” Any one of the multibase blocks can be used to represent one unit. Familiarize yourself with the multibase blocks by doing the following Learning Exercises and making up more problems until you feel you are familiar with the blocks and their relationships.

Example: This next sketch represents numbers with bases larger than five because there are five flats. If the little cube represents the unit one, the number here is 520 for any base larger than five. If the long represents one, then the number represented here is 52 in any base larger than five. If the flat represents one, then the number represented here is 5.2 in any base larger than five.

\[
\begin{array}{cccccc}
\square & \square & \square & \square & \square & \square \\
\end{array}
\]

Discussion: Representing numbers with multibase notation

Here is a representation of a number: 

\[
\begin{array}{cccccc}
\square & \square & |||
\end{array}
\]
Which bases could use this representation? Why or why not? What are some possible numbers that can be represented by this drawing?

In base eight, how many small cubes are in a long? How many in a flat? How many in a large cube? How many longs in a flat? How many flats in a large cube? Answer the same questions for base ten; for base two.

One can also represent decimal numbers with base ten blocks or drawings. You must first decide which block represents the unit. If the unit is the long, then the small block is one-tenth, the flat is ten, and the large block is 100. Thus 2.3 in base ten could be represented as: \[ | \quad \square \quad \square \quad \square \]

**Activity: Representing Numbers With Multibase Blocks**

For these problems, use your cutout blocks (in the appendix) or use drawings such as shown above. Note that the drawings do not show the markings of the base that appears in the picture of the blocks, and thus do not clearly indicate the base in the way that multibase blocks do.

1. Represent 2.3 in base ten using the small cube as one unit.
   Represent 2.3 using another cube as the unit. Compare your representation with a neighbor.

2. Use the base five blocks to represent 2.41\_five in two different ways.
   Be sure to indicate which piece represent the unit in each case.

Take-Away Message...We could just as easily have based our number system on something other than ten, but ten is a natural number to use because we have ten fingers. By working in bases other than ten, you have probably gained a new perspective on the structure and complexity of our place value system, particularly the importance of the value of each place. This understanding unifies all of the procedures we use in calculating with numbers in base ten. As teachers, you will need this knowledge to help students understand computational procedures.

**Learning Exercises for Section 2.3**

1. If you have access to it, use the BasePlayer applet to practice counting in different bases.
2. Write ten (this many: 

   \[\begin{array}{cccccc}
   1 & 1 & 1 & 1 & 1 & 1 \\
   \end{array}\]

) in each of these systems.
   \begin{enumerate}
   \item base four
   \item base five
   \item base eight
   \end{enumerate}

3. Write each of these.
   \begin{enumerate}
   \item four in base four
   \item eight in base eight
   \item twenty in base twenty
   \item b in base b
   \item b^2 in base b
   \item b^3 + b^2 in base b
   \item twenty-nine in base three
   \item one hundred fifteen in base five
   \item 69_{ten}, in base two
   \item 1728_{ten}, in base twelve
   \end{enumerate}

4. Write the numerals for counting in base two, from one through twenty.

5. How do you know that there is an error in each of these?
   \begin{enumerate}
   \item ten = 24_{three}
   \item fifty-six = 107_{seven}
   \item thirteen and three-fourths = 25.3_{four}
   \end{enumerate}

6. Write each of these in the usual base ten words and as a base ten numeral. For example, \(111_{two} = (1 \times 2^2) + (1 \times 2) + (1 \times 1) = 7_{ten}\) and \(31.2_{four} = (3 \times 4) + (1 \times 1) + \frac{2}{4} = 12 + 1 + \frac{1}{5} = 13.5\).
   \begin{enumerate}
   \item 37_{twelve}
   \item 37_{nine}
   \item 207.0024_{ten}
   \item 1000_{two}
   \item 1,000,000_{two}
   \item 221.2_{three}
   \end{enumerate}

7. For a given number, which base—two or twelve—will usually have a numeral with more digits? What are the exceptions?

8. In what bases would \(4025_b\) be a legitimate numeral?

9. Compare these pairs of numbers by placing < or > or = in each box.
   \begin{enumerate}
   \item 34_{five} \[\square\] 34_{six}
   \item 4_{five} \[\square\] 4_{six}
   \item 43_{five} \[\square\] 25_{six}
   \item 100_{five} \[\square\] 18_{nine}
   \item 111_{two} \[\square\] 7_{ten}
   \item 23_{six} \[\square\] 23_{five}
   \end{enumerate}

10. On one of your space voyages, you uncover an alien document in which some “one, two,...” counting is done: obi, fin, mus, obi na, obi obi, obi fin, obi mus. What base does this alien civilization apparently use? Continue counting through twenty in that system.

11. Hints of the influence of other bases remain in some languages. What base could have led to each of these?
a. French for eighty is “quatre-vingt.”

b. The Gettysburg Address, “Four score and seven years ago…”

c. A gross is a dozen dozen.

d. A minute has 60 seconds, and an hour has 60 minutes.

12. What does $34.2_{\text{five}}$ mean? What is this number written in base ten?

13. In each, write the “basimal” place values and then the usual base ten fraction or mixed number. Here is an example:

$$10.111_{\text{two}} = (2 + 0 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8})_{\text{ten}} = 2 \frac{7}{8} \text{ (Note: } 4 = 2^2 \text{ and } 8 = ?)$$

a. $21.23_{\text{four}}$  
  b. $34.3_{\text{twelve}}$

14. Write each of these in “basimal” notation.

Example: three-fourths in base ten is what in base two?

$$(\frac{3}{4})_{\text{ten}} = (\frac{1}{2} + \frac{1}{4})_{\text{ten}} = 0.11_{\text{two}}$$

a. one-fourth, in base twelve  
  b. three-fourths, in base twelve

c. one-fourth, in base eight

15. Give the base ten numeral for each of the following.

a. $101010_{\text{two}}$  
  b. $912_{\text{twelve}}$  
  c. $425_{\text{six}}$

  d. $41.5_{\text{eight}}$  
  e. $1341_{\text{five}}$

16. Write this many $\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$ (it is $12_{\text{ten}}$) in each of these bases.

a. nine  
  b. eight  
  c. seven  
  d. six  
  e. five  
  f. four  
  g. three  
  h. two

17. Write $100_{\text{ten}}$ in each of these bases.

a. seven  
  b. five  
  c. eleven  
  d. two  
  e. thirty-one

18. Complete with the proper digits.

a. $57_{\text{ten}} = \text{_____five}$  
  b. $86_{\text{nine}} = \text{_____ ten}$

c. $312_{\text{four}} = \text{_____ ten}$  
  d. $237_{\text{ten}} = \text{_____ eight}$

e. $2101_{\text{three}} = \text{_____ ten}$  
  f. $0.111_{\text{two}} = \text{_____ ten}$

19. Represent 34 in base ten, with the small block as the unit; with the long as the unit.
20. Represent \(234_{\text{five}}\) with the small cube as the unit; with the long as the unit. (Notice that 234 does not mean two-hundred thirty-four here.) Represent \(234_{\text{six}}\). (If you have only base ten blocks available, then sketch drawings for this exercise.)

21. In base six, 5413 is \(\underline{\underline{\text{ones}}}\), is \(\underline{\underline{\text{sixes}}}\), is \(\underline{\underline{\text{six}^2}}\); is \(\underline{\underline{\text{six}^3}}\)

22. Represent 2.34 in base ten with the flat as the unit.

23. Decide on a representation with base ten blocks for...

   a. 3542
   b. 0.741
   c. 11.11

24. Represent 5.4 and 5.21 with base ten blocks, using the same block as the unit. (What will you use to represent one?) Many school children say that 5.21 is larger than 5.4 because 21 is larger than 4. How would you try to correct this error using base ten blocks?

25. Someone said, “A number can be written in many ways.” Explain that statement.

\[2.4 \text{ Operations in Different Bases}\]

Just as we can add, subtract, multiply and divide in base ten, so can we perform these arithmetic operations in other bases. The standard algorithm for addition, depicted first below, is commonly used and is probably known to all of you. The expanded algorithms make the processes easier to understand. Once it is well understood, it is easily adapted to become the standard algorithm. Not all standard algorithms in this country are used in other countries, so the word “standard” is a relative one.

In base ten we could add 351 + 250 in these two ways. (There are other ways, of course.) The first way is called the standard algorithm, and is probably the one you were taught. It consists of thinking: \(1 + 0 = 1\), so place 1 in the one’s place in the answer. Next, 5 tens and 5 tens is 10 tens, which is 100, so place 1 in the hundreds column, place 0 in the ten’s place in the answer. Now there are 6 hundreds, so place 6 in the hundred’s place in the answer.
The second way, called an *expanded* algorithm is now being taught in some schools. Note how place value is attended to here, whereas in the standard algorithm, each “column” is treated the same: 1 + 0, 5 + 5, and 3 + 2 + 1. The expanded algorithm instead considers 1 + 0, 50 + 50, and 300 + 200. After the expanded algorithm is understood, it can be condensed into the standard procedure (called an algorithm).

\[
\begin{array}{c}
1 \\
351 \\
+ 250 \\
601 \\
\end{array}
\begin{array}{c}
351 \\
+ 250 \\
1 \text{ (thinking } 1 + 0) \\
100 \text{ (thinking } 50 + 50) \\
500 \text{ (thinking } 300 + 200) \\
601 \\
\end{array}
\]

Ask: How many ones? How many eights? How many 8 x 8 s?

We can also use either method for adding in other bases, but the second method is sometimes easier to follow until adding in another base is well understood.

**Example:** Here is an example using both the standard and expanded algorithms to add the same two numbers in base ten and base eight. Make sure you can understand each way in each base.

\[
\begin{array}{c}
1 \\
351_{\text{ten}} \\
+ 250_{\text{ten}} \\
601_{\text{ten}} \\
\end{array}
\begin{array}{c}
351_{\text{ten}} \\
+ 250_{\text{ten}} \\
1_{\text{ten}} \text{ thinking } (1 + 0) \\
100_{\text{ten}} \text{ thinking } (50 + 50) \\
500_{\text{ten}} \text{ thinking } (300 + 200) \\
621_{\text{ten}} \text{ thinking } (100 + 500) \\
\end{array}
\]

\[
\begin{array}{c}
1 \\
351_{\text{eight}} \\
+ 250_{\text{eight}} \\
621_{\text{eight}} \\
\end{array}
\begin{array}{c}
351_{\text{eight}} \\
+ 250_{\text{eight}} \\
1_{\text{eight}} \text{ thinking } (1 + 0)_{\text{eight}} \\
120_{\text{eight}} \text{ thinking } (50 + 50)_{\text{eight}} \\
500_{\text{eight}} \text{ thinking } (300 + 200)_{\text{eight}} \\
621_{\text{eight}} \text{ thinking } (100 + 500)_{\text{eight}} \\
\end{array}
\]

**Activity: Adding in Base Four**

Add these two numbers in base four in both expanded and standard algorithms: 311_{four} and 231_{four}.
If we can add in different bases, we should be able to subtract in different bases. Here is an example of how to do this.

**Example:** Find $321_{five} - 132_{five}$. One way to think about this problem is to regroup in base five just as we do in base ten, then use the standard way of subtracting in base ten.

**Solution:**

\[
\begin{array}{c}
  321 \\
- 132 \\
\end{array}
\]

**Step 1:** We cannot remove 2 ones from 1 one, so we need to take one of the fives from $321_{five}$ and change it to five ones:

\[321_{five} \rightarrow 300_{five} + 20_{five} + 1_{five} \rightarrow 300_{five} + 10_{five} + 11_{five}\]

**Step 2:** We can now take 2 ones from 11 ones (in base five) leaving 4 ones. (Notice how 321 has changed with 3 five squared, then 1 five, then 11 ones, from Step 1.)

\[
\begin{array}{c}
  1 \\
- \begin{array}{c}
  3 \\
  2 \\
\end{array} \begin{array}{c}
  1 \\
  2 \\
\end{array}_{five} \\
\end{array}
\]

This means 3 (five squared) + 1 five + 11 ones as in Step 1.

**Step 3.** In the fives place: We cannot subtract 3 fives from 1 five, so we must change one five squared to five sets of five. This, together with the one five already in place, we know have 11 fives.

That is: $300_{five} + 10_{five} \rightarrow 200_{five} + 110_{five}$, so

\[
\begin{array}{c}
  2 \\
- \begin{array}{c}
  3 \\
  2 \\
\end{array} \begin{array}{c}
  1 \\
  2 \\
\end{array}_{five} \\
\end{array}
\]

This means 11 fives, not 11 ones, so the 11 stands for 110 and 11 fives minus 3 fives is 3 fives, or $110 - 30$ is 30).

\[
\begin{array}{c}
  1 \\
- \begin{array}{c}
  3 \\
  4 \\
\end{array}_{five} \\
\end{array}
\]

We now have 2 (five squared) from which 1 (five squared) is subtracted, leaving 1 (five squared). The answer is 1 five-squared plus 3 fives plus 4 ones which is $134_{five}$.

**Activity: Subtracting in Base Four**

*Subtract $231_{four}$ from $311_{four}$ in base four.*
We can also multiply and divide in different bases. However, the intent here is to introduce you to different bases so that you have a better understanding of our own base ten system, and that you understand why children need time to learn to operate in base ten. Thus there are no examples or exercises provided here for multiplication and division in different bases, although it is certainly possible to carry out these operations.

We can use base materials help visualize adding and subtracting in other bases. You can cutout and use materials from the appendix on bases.

Example. Suppose we want to add $23_{\text{four}}$ and $311_{\text{four}}$ using base four blocks, using the small block as the unit. We could first express the problem as

```
\[
\begin{array}{c}
\underline{\text{2}} \underline{\text{3}} \underline{\text{0}} \\
\underline{\text{3}} \underline{\text{1}} \underline{\text{1}}
\end{array}
\]
```

We have too many longs (in base four), so trade four longs for a flat. Now we have too many flats (each representing four squared). Trade four flats for a large cube (which represents four cubed).

```
\[
\begin{array}{c}
\text{4} \text{4} \\
\text{4} \text{4} \\
\text{4} \text{4}
\end{array}
\]
```

Represents the answer, which is $1202_{\text{four}}$

The blocks represent one four cubed, two four squared, and two ones.

Example: Suppose we want to subtract $23_{\text{four}}$ from $32_{\text{four}}$. This time let us use the long as my unit, $32_{\text{four}}$ is represented:

```
\[
\begin{array}{c}
\underline{\text{3}} \underline{\text{2}} \\
\underline{\text{2}} \underline{\text{3}}
\end{array}
\]
```

I cannot remove 3 longs (ones) until I change a flat to four longs (which means change one nine into three threes)

```
\[
\begin{array}{c}
\underline{\text{2}} \underline{\text{2}} \underline{\text{1}} \\
\underline{\text{1}} \underline{\text{1}} \underline{\text{1}}
\end{array}
\]
```

Remove two flats; three longs

```
\[
\begin{array}{c}
\text{3} \text{1} \\
\text{3} \text{1}
\end{array}
\]
```

To take away $23_{\text{four}}$ we must remove three longs (three ones), and two flats (two threes), and we are left with $3_{\text{four}}$ as the difference.

Think About...If we had used the small block as the unit in the above subtraction example, would the answer be different? Try it.
Take-Away Message... Arithmetic operations in other bases are undertaken in the same way as in base ten. However, because we have less familiarity with other bases, arithmetic operations in those bases take longer than operations in base ten. For children not yet entirely familiar with base ten, time needed to complete arithmetic operations takes longer than it does for us.

**Learning Exercises for Section 2.4**

1. Add $11111_{three}$ and $20102_{three}$ each of the ways illustrated above.
   Which way did you find it easier?

2. Do these exercises in the designated bases, using MultiBase Blocks (or the cardboard cutouts) and by using the software Base-ic Addition. Be SURE to designate which block represents the unit.
   
   a. $341_{five}$  
   b. $101_{two}$  
   c. $321_{four}$  
   d. $296_{ten}$  
   
   + $220_{five}$  
   + $110_{two}$  
   - $123_{four}$  
   - $28_{ten}$

3. Add the following in the appropriate bases, without Blocks unless you need them.

   a. $2431_{five}$  
   b. $351_{nine}$  
   c. $643_{seven}$  
   d. $99_{eleven}$  
   
   + $223_{five}$  
   + $250_{nine}$  
   + $134_{seven}$  
   + $88_{eleven}$

4. Subtract in different bases, , without Blocks unless you need them.

   a. $351_{nine}$  
   b. $643_{seven}$  
   c. $2431_{five}$  
   d. $772_{eleven}$  
   
   - $250_{nine}$  
   - $134_{seven}$  
   - $223_{five}$  
   - $249_{eleven}$

5. Do you think multiplying and dividing in different bases would be difficult? Why or why not?

6. Use the cut-outs from an appendix for the different bases to act out the following. As you act each out, record what would take place in the corresponding numerical work.

   a. $232_{four}$  
   b. $232_{five}$  
   c. $232_{eight}$  
   d. $101_{two}$  
   
   $13_{five}$  
   $13_{eight}$  
   $11_{two}$
   
   $113_{four}$  
   $113_{five}$  
   $113_{eight}$  
   $111_{two}$
7. Use the cut-outs from an appendix for the different bases to act out the following. As you act each out, record what would take place in the corresponding numerical work.

\[
\begin{align*}
&\quad \text{a. } 200_{\text{six}} \quad \text{b. } 200_{\text{five}} \quad \text{c. } 200_{\text{eight}} \quad \text{d. } 100_{\text{two}} \\
&\quad -13_{\text{four}} \quad -13_{\text{five}} \quad -13_{\text{eight}} \quad -11_{\text{two}}
\end{align*}
\]

8. Describe how cut-outs for base six would look. For base twelve.

2.5 Issues for Learning: Understanding Place Value

The notion that ten ones and one ten give the same number is vital to understanding the usual numeration system, as are the later rethinking of ten tens as one hundred, ten hundreds as one thousand, etc. Understanding place value is considered to be foundational to elementary school mathematics.

But base ten for children might be as mysterious as base b may have been for you. (Admittedly, your extensive experience with base ten also gets in the way!) By working with other bases, you have had the opportunity to explore what it means to have a place-value system where each digit has a particular meaning, and thus come to a better understanding of our base ten system of writing numbers and calculating with numbers.

One activity-centered primary program incorporates many activities involving grouping by twos, by threes, and so on, even before extensive work with base ten groupings, to accustom the children to counting not just one object at a time, but groups each made up of several objects. Ungrouping needs to also be included. That is, 132 could be regarded as one one-hundred, three tens, and 2 ones. Or, it could be regarded as one one-hundred and 32 ones. Here, the 3 tens are “unbundled” to make 30 ones. Regarding a group made up of several objects as one thing is a major step that needs instructional attention.

The manner in which we vocalize numbers can sometimes cause problems for students. For example, some young U.S. children will write 81 for eighteen, whereas scarcely any Hispanic children (diez y ocho = eighteen) or Japanese children (ju hachi = eighteen) do so. (Some wishfully think
we should say “onetyeight” for eighteen in English.) What other numbers can cause the same sort of problem that eighteen does?

Place value instruction in schools is often superficial and limited to studying only the placement of digits. Thus, children are taught that the 7 in 7,200 is in the thousands place, the 2 is in the hundreds place, a 0 is in the tens place, and a 0 is in the ones place. But when asked how many hundred dollar bills could be obtained from a bank account with $7,200 in it, or how many boxes of ten golf balls could be packed from a container with 7,200 balls, children almost always do long division, dividing by 100 or by 10. They do not read the number as 7200 ones, or 720 tens, or 72 hundreds, and certainly not as 7.2 thousands. But why not? These are all names for the same number, and the ability to rename in this way provides a great deal of flexibility and insight when working with the number. (It is interesting that we later expect students to understand newspaper figures such as $3.2 billion. What does .2 billion mean here?)

Over the years many different methods have been used to teach place value. The abacus with nine beads on each string is one type of device used to represent place value. The Base Ten Blocks pictured in Section 2.3 have been extensively used to introduce place value and operations on whole numbers and decimal numbers. One problem with these representations, however, is that students do not always make the connections between what is shown with the manipulative devices and what they write on paper.

Our place value system of numeration extends to numbers less than 1 also. The naming of decimal numbers needs special attention. The place value name for 0.642 is six hundred forty-two thousandths. Compare this to reading 642, where we simply say six hundred forty-two, not 642 ones. This is a source of confusion that is compounded by the use of the tenths or hundredths with decimal numbers, the use of ten or hundred with whole numbers, and the additional digits in the whole number with a similar name. The number 0.642 is read 642 thousandths, meaning 642 thousandths of one, while 642,000 is read 642 thousand, meaning 642 thousand ones. That tens and tenths, hundreds and hundredths, etc., sound so much alike no doubt causes some children to lose sense-making when it
comes to decimals. Some teachers resort to a digit-by-digit pronunciation—"two point one five" for 2.15—but that removes any sense for the number; it just describes the numeral. Plan to give an artificial emphasis to the -th sound when you are discussing decimals with children. (You can also say "decimal numeral two and three-tenths" and "mixed numeral two and three-tenths" to distinguish 2.3 and 2 $\frac{3}{10}$.)

To compare 0.45 and 0.6, students are often told to "add a zero so the numbers are the same size." (Try figuring out what this might mean to a student who does not understand decimal numbers in the first place!) The strategy works, in the sense that the student can then (usually) choose the larger number, but since it requires no knowledge of the size of the decimal numbers, it does not develop understanding of number size. Instead of annexing zeros, couldn't we expect students to recognize that six-tenths is more than forty-five hundredths because 45 hundredths has only 4 tenths and what is left is less than another tenth? But for students to do this naturally, they must have been provided with numerous opportunities to explore—and think about—place value. Comparing and operating on decimals, if presented in a non-rule oriented fashion, can provide these opportunities. If teachers postpone work with operations on decimals until students conceptually understand these numbers, students will be much more successful than if teachers attempt to teach computation too early. Some researchers have shown that once students have learned rote rules for calculating with decimals, it is extremely difficult for them to relearn how to calculate with decimals meaningfully.

2.6 Check Yourself

In this chapter you have explored the ways we express numbers. Historically, there were many numeration systems used to express numbers in different ways. A place value numeration system such as the modern world now uses provides a far more efficient way to express numbers than ancient systems, such as the Roman numeral system. Our use of base ten is probably due to the fact that we have ten fingers. Other bases could be used. Because we are so familiar with base ten, however, working with other bases is useful in appreciating the difficulties children
have in learning to use base ten, particularly when operating with numbers in base ten.

Understanding place value and its role in the elementary school mathematics curriculum is crucial. Too many teachers think that teaching place value is simply a matter of noting which digit is in the ones place, which is in the tens place, etc. But it is only when students have a deep understanding of place value that they can make sense of numbers larger than 10 and smaller than 1, and understand how to operate on these numbers. Most arithmetic errors (beyond careless errors) are in fact errors due to a lack of understanding of place value. Unfortunately, the algorithms we teach usually treat digits in columns without attending to their values, and students who learn these algorithms without understanding the place value of each digit are far more likely to make computational errors.

You should be able to work problems like those assigned and to do the following:

1. Discuss the advantages of a place value system over other ancient numeration systems.

2. Explain how the placement of digits determine the value of a number in base ten, on both sides of the decimal point.

3. Explain how the placement of digits determine the value of a number in any base, such as base five or base twelve and answer questions such as: What does 346.3 mean in base twelve? Convert that number to base ten.

4. Given a particular base, write numbers in that system beginning with one.

5. Make a drawing that demonstrates a particular addition or subtraction problem, e.g., 35.7 + 24.7 or 35.7 - 24.7 in base ten.

6. Write base ten numbers in another base, such as 9 in base nine, or 33 in base two.
7. Add and subtract in different bases.

8. Understand the role of one in reading and understanding decimal numbers.

9. Discuss problems that children who do not have a good understanding of place value might have when they do computation problems.

References for Chapter 2


